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DISCUSSION

Analytical study of induced anisotropy in idealized granular materials

L. ROTHENBURG and R. J. BATHURST (1989). *Géotechnique* **39**, No. 4, 601–614

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The Authors present interesting data obtained from computer simulated experiments on an assembly of discs. Their results and observations are in general agreement with previously published work. However, as stated in the Paper, the main objective is to establish mathematical relationships between the internal contact forces, contact orientations and the ensemble average of the stress tensor. This has been the subject of a number of papers, e.g. Mehrabadi *et al.* (1982), Cundall & Strack (1983), Thornton & Barnes (1986). What distinguishes the Paper from previous publications is the explicit use of probability density functions expressed in terms of Fourier series and the fact that the derived relationships between stress, structure and contact forces differ from previous solutions.

An initial reading of the Paper suggests that, since the coefficient $\bar{f}_0$ is defined as the average normal force over all contacts in the assembly (p. 608), the relationship for the centre of the Mohr circle (equation (19)) is inconsistent with equation (16) and should be

$$(\sigma_{11} + \sigma_{11})/2 = (M_v l_0 \bar{f}_0)/2 \qquad (22)$$

in agreement with Thornton & Barnes (1986). However, in a later paper, Bathurst & Rothenburg (1989), state that $\bar{f}_0$ is not the average force computed over all contacts in the assembly but is a 'mean over any subset of contacts selected in such a way that the distribution of the subset is isotropic even though the distribution of the entire assembly is not.' They go on to explain that it is due to this definition that the isotropic stress is a function of the degree of fabric and contact force anisotropy. It is noted that equation (6) defines $\bar{f}_0$ as the overall average of the average forces f_n for each group orientation.

Since Thornton & Barnes (1986) adopted a somewhat different averaging procedure it is of interest to identify the corresponding probability density functions and resulting expressions for the invariants of the stress tensor. For the purpose of this exercise it is sufficient to consider only the normal force contribution to the stress tensor and hence we assume an assembly of frictionless discs. For frictionless discs equation (13) may be written, using the notation of Thornton & Barnes (1986), in the form

$$\sigma_{ij} = (1/V) \sum \boldsymbol{R}_i \boldsymbol{N}_j \quad i, j = 1, 2 \qquad (23)$$

where $\boldsymbol{N}_j$ is the normal force vector acting at the contact defined by the radius vector $\boldsymbol{R}_i$ and the summation is performed over the M contacts (each contact is counted twice) occupying a volume V. Since $\boldsymbol{R}_i = R\boldsymbol{n}_i$ and $\boldsymbol{N}_j = N\boldsymbol{n}_j$, where $\boldsymbol{n}_i$ defines the contact normal vector, equation (23) becomes

$$\sigma_{ij} = (1/V) \sum RN\boldsymbol{n}_i \boldsymbol{n}_j \qquad (24)$$

or

$$\sigma_{ij} = (2M/V)\langle RN\boldsymbol{n}_i \boldsymbol{n}_j\rangle \qquad (25)$$

where the brackets ⟨ ⟩ denote statistical average. Barnes (1985) demonstrated, using data from computer simulated experiments on disc assemblies, that

$$\langle RN\boldsymbol{n}_i \boldsymbol{n}_j\rangle = \langle R\rangle\langle N\boldsymbol{n}_i \boldsymbol{n}_j\rangle \qquad (26)$$

Hence we may write

$$\sigma_{ij} = (2M\boldsymbol{R}/V)\langle N\boldsymbol{n}_i \boldsymbol{n}_j\rangle \qquad (27)$$

or, more conveniently,

$$\sigma_{ij} = (2MRN/V)\langle N\boldsymbol{n}_i \boldsymbol{n}_j/N\rangle \qquad (28)$$

The statistical average term in equation (28) indicates that it is necessary to consider a weighted distribution of the contact normal vectors since

$$\langle N\boldsymbol{n}_i \boldsymbol{n}_j\rangle/\langle N\rangle = \sum N\boldsymbol{n}_i \boldsymbol{n}_j \Big/ \sum N \qquad (29)$$

Returning to equation (27), we may consider the volume to be infinite and express the stress tensor in terms of a probability density function $N(\theta)$ since

$$\langle N\boldsymbol{n}_i \boldsymbol{n}_j\rangle \equiv \int N(\theta)\boldsymbol{n}_i \boldsymbol{n}_j \,\mathrm{d}\theta \qquad (30)$$

where

$$\int N(\theta)\,\mathrm{d}\theta = N \qquad (31)$$

and N is the average contact force over all contacts in the assembly. To avoid unnecessary complications we will now assume that the reference frame axes coincide with the principal anisotropy axes and adopt a second-order Fourier representation for $N(\theta)$. We define the probability density function as

$$N(\theta) = (N/2\pi)(1 + N_2 \cos 2\theta) \quad (32)$$

and equation (27) therefore becomes

$$\sigma_{ij} = (2MRN/V)(1/2\pi) \times \int (1 + N_2 \cos 2\theta)\boldsymbol{n}_i \boldsymbol{n}_j \, d\theta \quad (33)$$

Considering the major principal stress σ_1 we may write

$$(V\sigma_1/2MR) = \langle N\boldsymbol{n}_1{}^2 \rangle = (N/2\pi) \int (1 + N_2 \cos 2\theta) \times \cos^2 \theta \, d\theta \quad (34)$$

Therefore

$$2\pi \langle N\boldsymbol{n}_1{}^2/N \rangle = \int \cos^2 \theta \, d\theta + (N_2/2) \times \int \cos 2\theta (1 + \cos 2\theta) \, d\theta \quad (35)$$

from which the definition of the Fourier coefficient N_2 is found to be

$$(N_2/4) = \langle N\boldsymbol{n}_1{}^2/N \rangle - 1/2 \quad (36)$$

From equations (34) and (36)

$$\sigma_1 = (2MRN/V)(1/2 + N_2/4) \quad (37)$$

Similarly, it can be shown that

$$\sigma_2 = (2MRN/V)(1/2 - N_2/4) \quad (38)$$

Hence

$$\sigma_1 + \sigma_2 = 2MRN/V \quad (39)$$

and the mobilized angle of internal shearing resistance is

$$(\sigma_1 - \sigma_2)/(\sigma_1 + \sigma_2) = N_2/2 \quad (40)$$

In three dimensions it is more convenient to use a tensorial representation for the probability density function, viz

$$N(\theta) = (N/2\pi)(1 + N_{ij} f_{ij}) \quad (41)$$

where

$$f_{ij} = \boldsymbol{n}_i \boldsymbol{n}_j - \delta_{ij}/3 \quad (42)$$

and

$$N_{ij} = 4 \int N(\theta) f_{ij} \, d\theta = 4\langle N f_{ij}/N \rangle \quad (43)$$

The first invariant of the stress tensor is then defined as

$$\sigma_{kk} = 2MRN/V \quad (44)$$

and the deviatoric stress tensor

$$s_{ij} = \sigma_{ij} - \sigma_{kk} \delta_{ij}/3 = \sigma_{kk}(N_{ij}/4) \quad (45)$$

Using the averaging procedures adopted by Thornton & Barnes (1986) we see, from above, that this leads to simple direct relationships between the Fourier coefficients and the deviatoric components of the statistical averages that characterize the weighted distribution of contact normal vectors. The relationship for the isotropic part of the stress tensor is much simpler than that of the Authors and although the deviatoric components of anisotropy are not simply additive, as in the Authors' formulation, equations (43) and (45) do reflect the fact that the magnitudes of the contact normal forces are correlated with the orientation of the contact normal vectors, as demonstrated by both computer simulated experiments and physical experiments on photoelastic disc assemblies.

It is clear that the differences between the above stress–structure–force relationships and those of the Authors are due to the different averaging procedures adopted. It will be interesting, therefore, to await results of more complex computer simulated experiments in order to find out which approach proves to be the more useful.

Authors' reply

Mr Thornton's discussion is related to the main result of the Paper, identification of shear strength components based on representation of the stress tensor as a weighted average of intergranular forces (equation (13)). He presented an alternative way of averaging microscopic contributions into the stress tensor and introduced a combined average characteristic of contact forces and fabric in terms of which the stress tensor can be expressed (equations (37)–(38)). Mr Thornton's relationship is somewhat simpler in representing the hydrostatic part of the stress tensor, while the expression given in the Paper is far more physically transparent in representing the deviatoric part of the stress tensor. The average characteristics of fabric and contact forces introduced in the Paper are such that shear strength of granular materials is represented in terms of additive contributions from contact force and fabric anisotropies (equation (8)).

As Mr Thornton pointed out, the differences between relationships given in the Paper and those presented in the discussion are due to the different averaging procedures adopted. The emphasis of the Paper, as its title suggests, is on effects of induced anisotropy. The advantage of relationships presented in the Paper, as we see it, is that it allows one to derive stress–strain relationships with minimal effort, as illustrated by Rothenburg *et al.* (1989a,b).

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