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NOTE ON A RANDOM ISOTROPIC GRANULAR MATERIAL WITH NEGATIVE POISSON'S RATIO

RICHARD J. BATHURST
Civil Engineering Department, Royal Military College of Canada, Kingston, Ontario, Canada K7K 5L0
and
LEO ROTHENBURG
Civil Engineering Department, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

Abstract—Poisson's ratio in generalized Hooke's law for an isotropic continuum is subject to the restriction $-1 \leq \nu \leq 0.5$. With the exception of recently developed low-density polymer foams, solids including granular materials have not been known to exhibit negative Poisson's ratio.

This paper is concerned with the verification of constitutive stress–strain relationships proposed by Rothenburg (Micromechanics of idealized granular systems. Ph.D. thesis, Carleton University, 1980) which describe macroscopic behaviour of idealized *bonded* granular materials by elastic parameters (K and ν) that are formulated explicitly in terms of microstructural parameters such as interparticle stiffness, contact density and average interparticle distance. The theory includes an expression for Poisson's ratio which is a function only of the ratio of tangential (shear) to normal contact stiffness λ. A negative Poisson's ratio is predicted for both planar and three-dimensional random isotropic systems when the tangential stiffness is greater than the normal stiffness (i.e. $\lambda > 1$). The results of numerical simulation of bonded disc assemblies used to verify constitutive relationships show that systems with $\lambda > 1$ do exhibit negative Poisson's ratio. Similar theoretical developments are summarized for three-dimensional random isotropic assemblies of bonded *spheres* and an analogous expression for Poisson's ratio is presented for these systems. It is noted that while a *negative* Poisson's ratio is theoretically possible, the micromechanical condition for $\lambda > 1$ is physically unlikely for particles of natural materials.

INTRODUCTION

The requirement that strain energy of a linear elastic isotropic solid must remain positive under any set of deformations results in well-known bounds on Poisson's ratio (i.e. $-1 \leq \nu \leq 0.5$). With the exception of certain recently developed low-density polymer foams [1], materials exhibiting a negative Poisson's ratio have not been observed even though negative values are theoretically admissible. This contradiction encourages the search for microstructure which can lead to a macroscopically-observed negative Poisson's ratio effect. In this paper it is shown that random isotropic assemblies of spherical particles interacting by means of forces transmitted through indestructable linear contacts would behave in this unusual manner when the ratio of shear to normal contact stiffnesses exceeds unity.

It appears that a material model similar to the one investigated in this paper was first proposed by the early founders of the theory of elasticity (i.e. Navier, Cauchy and Poisson) as a reflection of atomistic ideas in the 18th century. Molecules were viewed as particles interacting by means of long-range forces. Depending on the thoroughness of treatment and specific assumptions, the model admits a range of elastic stress–strain relationships with 1–21 independent elastic constants (e.g. [2]). Poisson's theory essentially assumes *next-neighbour* interactions as well as macroscopic isotropy and leads to the conclusion that the resistance to shear is 2/5 of Young's modulus [3]. In modern terminology this result means that Poisson's ratio is 1/4. Failure of this model to explain experimental data on many solids has eventually led to the concept of *bi-constant* isotropic materials [2].

Poisson's assumptions on next-neighbour central interactions and isotropy closely resemble microstructure of granular materials. From this point of view it is not surprising that the most widely quoted Poisson's ratio for sands is close to 1/4 although the separation of elastic and plastic strain in experiments on sand is difficult (e.g. [4]).

In the model considered in this paper, particles are considered to interact not only by means of normal contact forces (as in the Poisson's model) but, also by means of tangential forces. In essence, particles are considered *bonded* at contacts and contact force-interparticle compliance relationships are linear with independent normal and tangential (shear) stiffnesses. Isotropic linear elastic stress–strain relationships for this type of assembly were derived by Rothenburg

[5] and it was shown that the Poisson's ratio of the system depends on the ratio of interparticle tangential to normal stiffness λ. When λ is greater than unity, Poisson's ratio is negative. Furthermore, as λ approaches infinity Poisson's ratio tends to -1. In the present paper, this result is verified by numerically simulating a two-dimensional analogue of the three-dimensional system. The expression for Poisson's ratio for assemblies of bonded *discs* is somewhat different from the corresponding relationship of bonded spheres but all qualitative effects are preserved. Numerical simulations are also used to verify concepts and assumptions involved in the derivation of stress–strain relationships in terms of micromechanical parameters.

Results of this study suggest how a material with negative Poisson's ratio can be constructed artificially but also explain that natural materials with a negative Poisson's ratio are unlikely since the condition that contact tangential stiffness be greater than contact normal stiffness is improbable. Nevertheless, it is possible, as the recent work by Lakes (1987) on polymer foams has shown, that other types of microstructure may lead to a response characterized by a negative Poisson's ratio.

THEORETICAL BACKGROUND

Microstructural description

The planar assemblies under study are assumed to consist of a very large number of bonded discs. At the particle level, contact normals, contact vectors and contact forces can be associated with each physical contact as shown in Fig. 1. The orientation and magnitude of these quantities are assumed to be position independent within these systems (i.e. they are distributed homogeneously). The assembly microstructure (fabric) can be characterized by *average* coordination number γ and the orientation distribution of contact normals $E(\theta)$, [6] and contact lengths, $\bar{l}^c(\theta)$ as described by Rothenburg and Selvadurai [7] and Bathurst [8]. The coordination number is defined by $\gamma = M_V/N$ where M_V represents the total number of contacts within the assembly volume and N the total number of particles. The distribution function $E(\theta)$ can be used to determine the frequency of contact normals falling within the orientational class interval θ and $\theta + \Delta\theta$ for an infinite, statistically homogeneous, system according to:

$$M_g(\theta) = M_V \int_{\theta}^{\theta+\Delta\theta} E(\theta)\,d\theta \tag{1}$$

It should be noted that for any large but finite system of particles, distribution functions of the form (1) are a useful approximation.

In this paper, verification of fundamental constitutive relations is restricted to isotropic

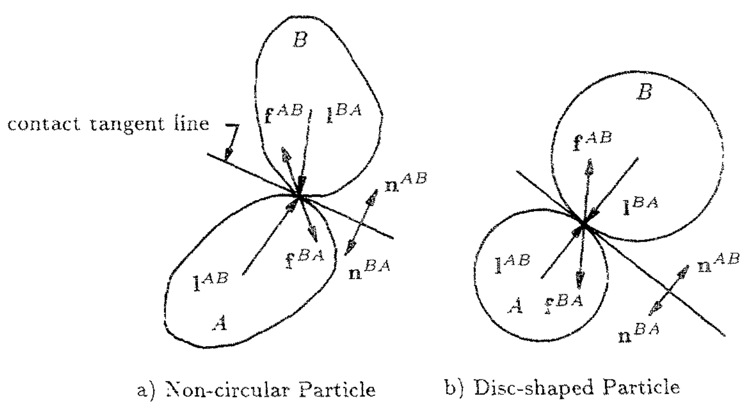

Fig. 1. Contact normals, contact vectors and contact forces.

assemblies defined by:

$$E(\theta)=\frac{1}{2\pi} \qquad (2)$$
$$\bar{l}^c(\theta)=\bar{l}_o$$

Here $\bar{l}_o$ represents the *average* contact length taken from all assembly contacts.

Average stress tensor

An average stress tensor in terms of the summation of discrete contact forces and fabric can be expressed as:

$$\bar{\sigma}_{ij}=\frac{1}{V}\sum_{c\in V} f_i^c l_j^c \quad i,j=1,2 \qquad (3)$$

Terms f_i^c and l_j^c refer to scalar components of contact forces $\mathbf{f}^c$ and contact vectors $\mathbf{l}^c$ at contact locations (refer to Fig. 1). Equivalent expressions for three-dimensional idealized granular assemblies have been reported by Weber [9], Dantu [10], Rothenburg [5], Christoffersen *et al.* [11] and Bathurst [8]. Rothenburg [5] and Rothenburg and Selvadurai [7] have proposed that expression (3) is a useful approximation to the stress tensor of continuum mechanics for granular assemblies comprising a large but finite number of particles. This equivalency can be understood by considering sums of force-contact vector components for many subregions of a given assembly volume. Quantities calculated from (3) would be expected to fluctuate from subvolume to subvolume. However, as the subdomains increase in volume and number of particles within a homogeneous system, these fluctuations can be expected to become smaller and smaller. This tendency to a single representative *average stress tensor* is assured by the composition of the function where each term makes a small contribution to

$$\frac{1}{V}f_i^c l_j^c.$$

For finite but large particle systems, the average stress tensor from discrete information is an accurate analogue to the stress tensor of continuum mechanics and in the following text they are assumed equivalent (i.e. $\sigma_{ij}=\bar{\sigma}_{ij}$).

An equivalent expression to (3) can be written by considering *group averages* $\overline{f_i^c l_j^c}(\theta_g)$ corresponding to a finite number of orientational class intervals. Specifically:

$$\sigma_{ij}=\frac{M_V}{V}\sum_{\theta_g}\overline{f_i^c l_j^c}(\theta)E(\theta)\,\Delta\theta \qquad (4)$$

Here a normalized discontinuous function $E(\theta)$ is used to describe the orientational distribution of contact normals. Assuming an *isotropic* assembly with

$$\lim_{V\to\infty},\ \lim_{M_V\to\infty} \text{ and } \lim_{\Delta\theta\to 0},$$

relation (4) can be expressed in integral form as:

$$\sigma_{ij}=\frac{m_v\bar{l}_o}{2\pi}\int_0^{2\pi}\bar{f}_i^c(\theta)n_j(\theta)\,\mathrm{d}\theta \qquad (5)$$

Here, the term $m_v=M_V/V$ is introduced for brevity and is used to denote contact density with respect to assembly area. The analogous relationship for three-dimensional *bonded spheres* can be recovered from relationships reported by Rothenburg [5], Rothenburg and Selvadurai

[7, 12] and Mehrabadi *et al.* [13]. The above expression forms the basis of a constitutive relationship for planar assemblies once the link between contact forces and strain is established.

Link between average forces and strain tensor

The link between contact forces and strain tensor can be made through a linear contact-force displacement law of the form:

$$\begin{aligned} f_n^c &= k_n \delta_n^c \\ f_s^c &= k_s \{\delta_t^c + \delta_w^c\} \end{aligned} \tag{6}$$

In these expressions, δ_n^c is the *relative* normal displacement at the contact which is measured with respect to the change in length between particle centres located a distance l apart. Terms δ_t^c and δ_w^c represent the contribution to relative tangential displacements at the contact due to relative translational movement of particle centres and relative particle rotations. The results of numerical simulations have shown that quantities δ_w^c are negligibly small for planar assemblies with γ approaching 6. For these assemblies, particle rotations are heavily constrained and normal and tangential forces (f_n^c, f_s^c) are due only to relative normal and translational displacements δ_n^c and δ_t^c.

It is reasonable to assume that if relative normal and tangential displacement components are averaged over orientational class intervals in the same manner as that described in the previous section then, *average* relative displacement components can be equated to *average* normal and tangential contact forces according to:

$$\begin{aligned} \bar{f}_n^c(\theta) &= k_n \bar{\delta}_n^c(\theta) \\ \bar{f}_s^c(\theta) &= k_s \bar{\delta}_t^c(\theta) \end{aligned} \tag{7}$$

Rothenburg [5] has proposed that average relative displacement components can be linked to strain tensor ϵ according to:

$$\begin{aligned} \bar{\delta}_n(\theta) &= \zeta(\epsilon_{ij} n_i n_j) \\ \bar{\delta}_t(\theta) &= \zeta(\epsilon_{ij} t_i n_j) \end{aligned} \tag{8}$$

Here $\mathbf{n} = (\cos\theta, \sin\theta)$, $\mathbf{t} = (-\sin\theta, \cos\theta)$ and the constant ζ is introduced as an unspecified coefficient of proportionality. From theoretical considerations Rothenburg [5] has proposed that for these systems $\zeta < 1$.

Constitutive relations for planar isotropic assemblies

Expression (5) can be rewritten to show the contribution of average normal contact forces and average tangential contact forces to average stress tensor:

$$\sigma_{ij} = \frac{m_v \bar{l}_o}{2\pi} \int_C^{2\pi} \{\bar{f}_n^c(\theta) n_i n_j + \bar{f}_s^c(\theta) t_i n_j\}\, \mathrm{d}\theta \tag{9}$$

Substitution of expressions (7) and (8) into (9) leads to constitutive relations of the form:

$$\sigma_{ij} = A_{ijkl} \epsilon_{kl} \quad i, j, k, l = 1, 2 \tag{10}$$

where:

$$A_{ijkl} = \frac{\zeta k_n \bar{l}_o m_v}{2\pi} \int_0^{2\pi} \{n_i n_j n_k n_l - \lambda t_i n_j t_k n_l\}\, \mathrm{d}\theta \tag{11}$$

In these expressions, parameter λ is introduced as the ratio of tangential to normal contact stiffness (i.e. $\lambda = k_s/k_n$).

Bulk and shear moduli for plane isotropic assemblies can be defined as:

$$\begin{aligned} \frac{\sigma_{11} + \sigma_{22}}{2} &= K(\epsilon_{11} + \epsilon_{22}) \\ \sigma_{11} - \sigma_{22} &= 2G(\epsilon_{11} - \epsilon_{22}) \\ \sigma_{12} &= 2G(\epsilon_{12}) \end{aligned} \tag{12}$$

Direct calculation of integrals defining components of A_{ijkl} results in equivalent bulk and shear moduli defined in terms of micromechanical parameters as follows:

$$K = \frac{m_v \bar{l}_o k_n \zeta}{4}, \qquad G = \frac{m_v \bar{l}_o k_n \zeta}{4}\left(\frac{1+\lambda}{2}\right) \tag{13}$$

Although the above moduli contain parameter ζ whose value is unspecified, Poisson's ratio is independent of ζ and depends only on the ratio of tangential to normal contact stiffness according to:

$$\nu = \frac{1-\lambda}{3+\lambda} \tag{14}$$

Examination of the above relationships shows that the admissible range of Poisson's ratio is $-1 \leq \nu \leq 1$ which is analogous to the admissible range in three-dimensional isotropic elastic continuum corresponding to $-1 \leq \nu \leq 1/2$. Relationship (14) shows that for the planar assemblies under study, a *negative* Poisson's ratio is possible if $\lambda > 1$.

Numerical simulation of bonded disc assemblies

General. In the current investigation, numerical simulation of assemblies comprising 1000 discs was undertaken to verify fundamental relationships proposed in the preceding text. A principal advantage of numerical simulation is that it allows all *microscopic* information to be extracted from the assemblies under study. In addition, the influence of micromechanical properties such as stiffness ratio λ can be assessed more readily from these experiments than from comparable physical models (i.e. photo-elastic disc assemblies).

Numerical simulations were carried out using a program called GLUE which is a heavily-modified version of the program BALL originally reported by Strack and Cundall [14] and used by them to investigate the micromechanical behaviour of *cohesionless* disc assemblies. Both programs implement a time-finite-difference scheme which solves the system of equations modelling a dynamic transient mechanical system. The mechanical system can be imagined as a network of lumped-mass-dashpot elements in which linear springs connect disc-shaped masses. Although the system is dynamic, the transient state approaches a static equilibrium condition if loading rates at the sample boundaries are kept low enough that inertial forces are always a small fraction of the average contact forces acting through the assembly. Kinetic energy is dissipated through the introduction of artificial damping, without which, the approximation to a static equilibrium condition would not be achieved. At the beginning of each calculation cycle, force components are applied to the centre of each boundary disc in response to prescribed boundary stress conditions.

Assembly generation

The assembly used in the current investigation was created by modifying a cohesionless assembly of discs first reported by Bathurst [8]. The modified assembly comprises a narrow range of disc diameters that have been *glued* together by imposing linear force-displacement laws of the form (6) to selected contacts. The resulting assembly is shown on Fig. 2 and has a (maximum) coordination number of $\gamma = 6$ and near-isotropic microstructure. The degree of microstructural anisotropy or orientational bias in contact normals can be assessed from the value of second and fourth-order terms in a Fourier series expression for the distribution function $E(\theta)$, [5]. The Fourier series function can be expressed as:

$$E(\theta) = \frac{1}{2\pi}\{1 + a\cos 2(\theta - \theta_a) + b\cos 4(\theta - \theta_b)\} \tag{15}$$

An analytical technique which can be used to extract the value of *coefficients of anisotropy* a and b and *directions of anisotropy* θ_a and θ_b from assembly data has been reported by Bathurst [8].

The microstructure for all assemblies in the current study is illustrated by the polar histograms plotted on Figs. 3 and 4. Figure 3 shows that these systems are essentially isotropic with respect to the *second* and *fourth-order* distribution of contact normals (i.e. a, $b \approx 0$). By

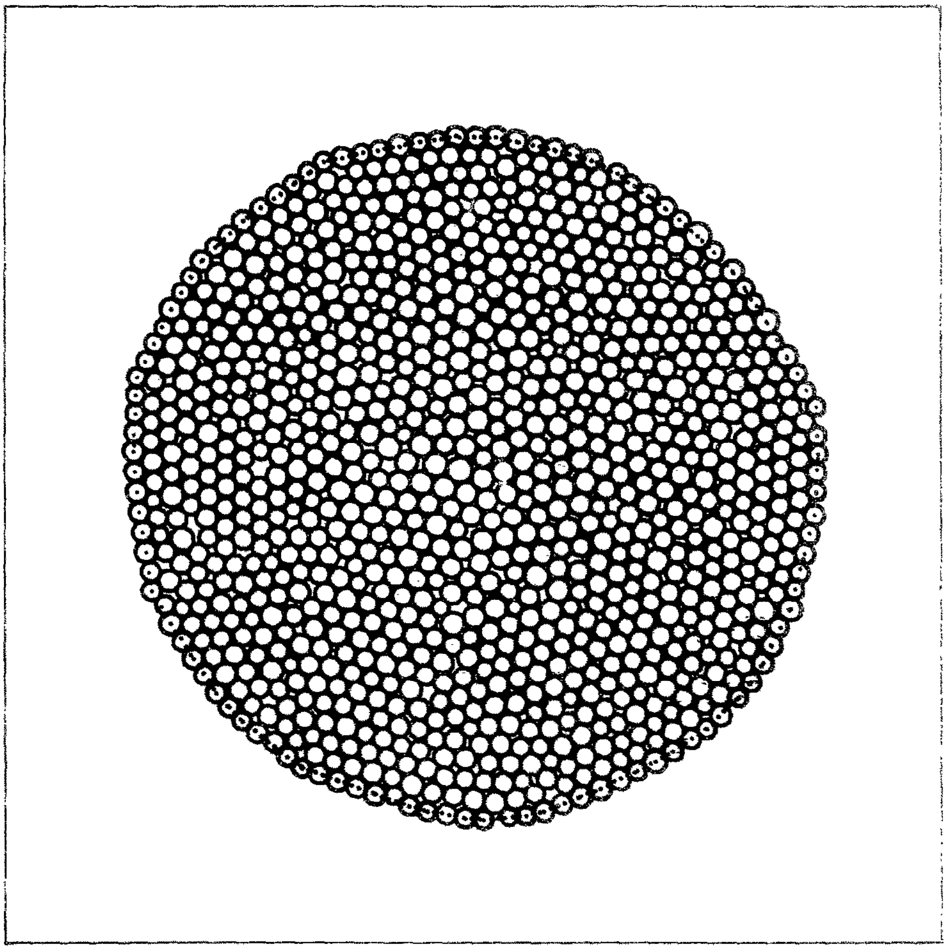

Fig. 2. 1000 disc assembly.

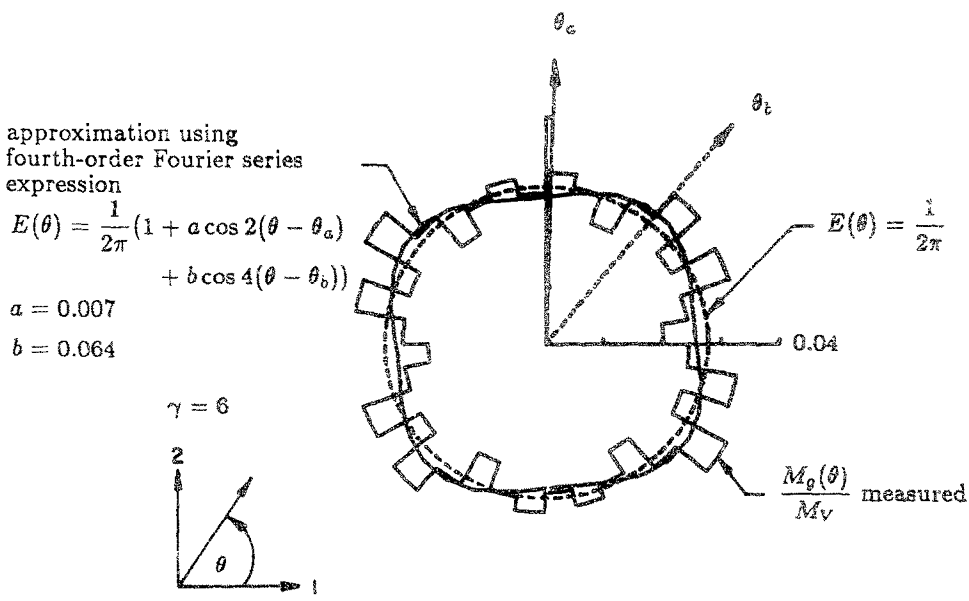

Fig. 3. Normalized distribution of contact orientations for 1000 disc assembly.

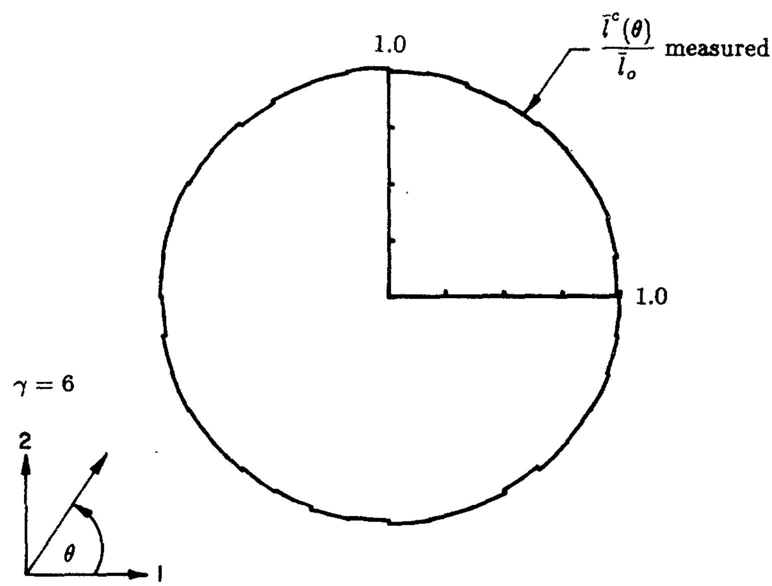

Fig. 4. Normalized distribution of contact lengths for 1000 disc assembly.

comparison, Bathurst [8] has generated anisotropic assemblies with coefficients of anisotropy as great as $a = 0.40$ and $b = 0.30$ by subjecting the same *cohesionless* assembly to large shearing deformations. Isotropic microstructure with respect to the distribution of contact lengths is clearly evident from Fig. 4.

Test program

A series of numerical simulations was undertaken to verify fundamental assumptions leading to bulk and shear moduli expressions and Poisson's ratio in terms of microstructural parameters.

Bonded assemblies comprising 1000 discs were subjected to biaxial compression by imposing at the sample boundary discrete forces approximating the stress state $\sigma_{22} > 0$, $\sigma_{11} = 0$ and $\sigma_{12} = \sigma_{21} = 0$. Under these conditions Poisson's ratio could be determined by calculating the resulting principal strain ratio from displacements recorded at the sample boundary.

Disc interactions in this investigation were controlled by the linear contact force-displacement laws given in expressions (6). The ratio of interparticle stiffnesses was kept constant for all contacts but was varied between tests over the range $0 \leq \lambda \leq 3$.

Tests results

A fundamental concept which allows the link between stress and strain to be made through micromechanical considerations is contained in relationships (8). Implicit in these expressions is the assumption that ζ is a direction-independent constant. Figures 5a and 5b plot ratios $\bar{\delta}_n^c(\theta)/(\epsilon_{ij}n_in_j)$ and $\bar{\delta}_t^c(\theta)/(\epsilon_{ij}t_in_j)$ as polar histograms. The data indicate that ζ values are sensibly direction-independent and nearly equal. This fundamental parameter is examined further in Fig. 6 by plotting ζ against a range of particle stiffness ratios. The figure shows that the average ζ value calculated from all contacts may be considered to be sensibly equivalent over the range $0 \leq \lambda \leq 3$.

Superimposed on Fig. 6 are ζ values calculated from measured values for bulk and shear moduli K and G. A similar range of values for ζ results from this exercise as that determined directly from measured contact distortions. The figure illustrates that within a small error margin, *microscale* information and fundamental relationships (13) predict similar values for the ζ coefficient. A reasonable representative value for ζ from these numerical simulations is 0.94. The small deviations from this value are considered to be the result of stress-induced microstructural anisotropy which is sensitive to the magnitude of interparticle stiffness k_n and k_s.

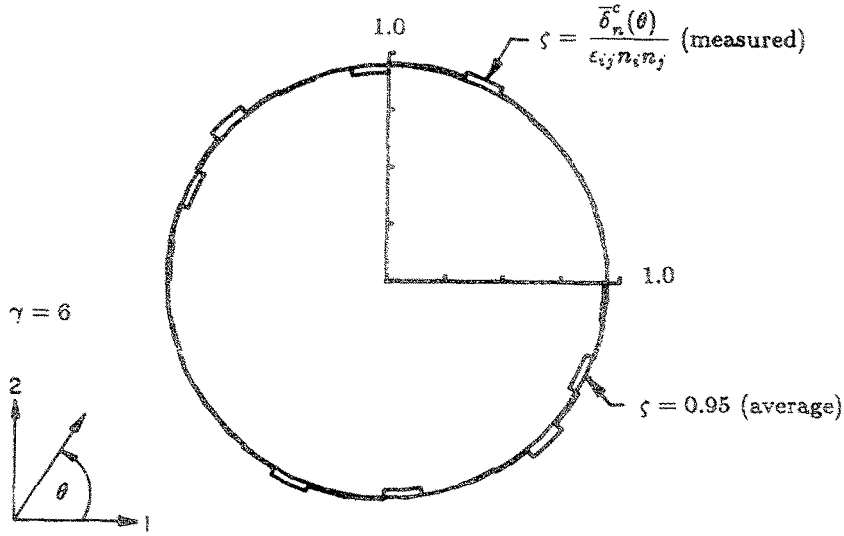

Fig. 5a. Coefficient ζ from distribution of average normal contact displacements $\bar{\delta}_n^c(\theta)$.

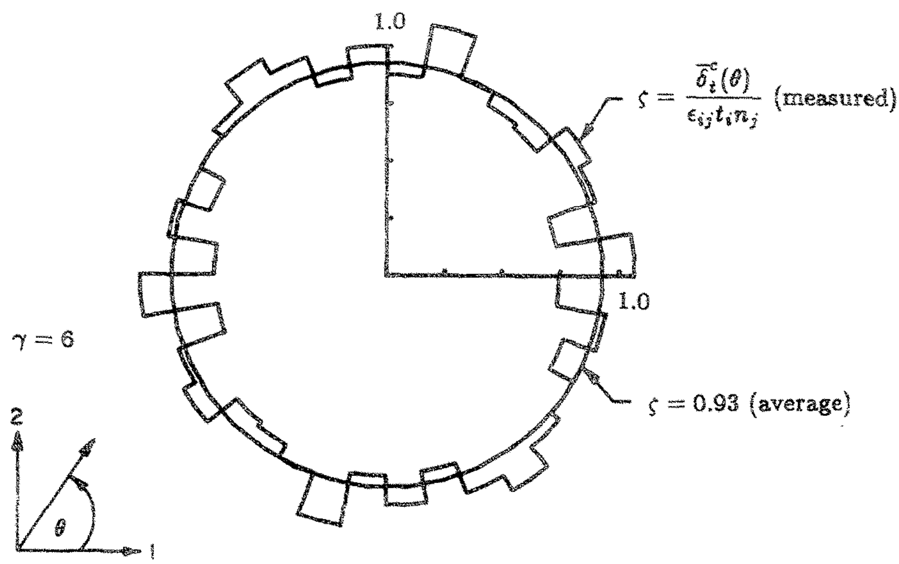

Fig. 5b. Coefficient ζ from distribution of average tangential contact displacements $\bar{\delta}_t^c(\theta)$.

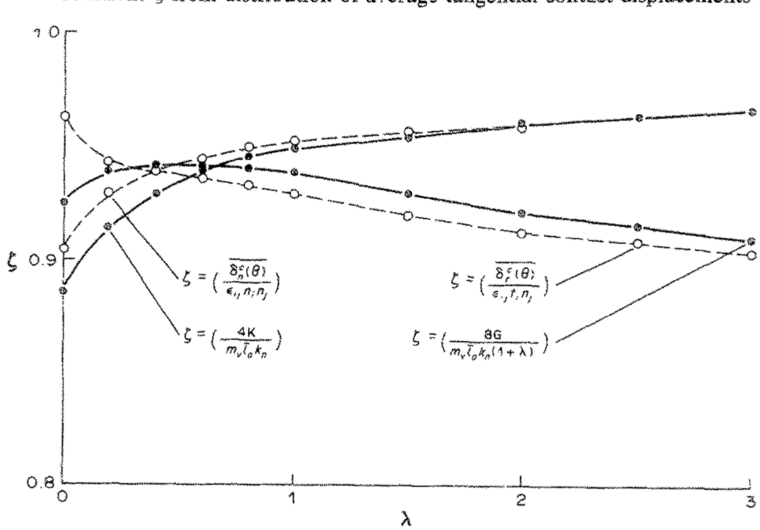

Fig. 6. Coefficient ζ versus contact stiffness ratio λ.

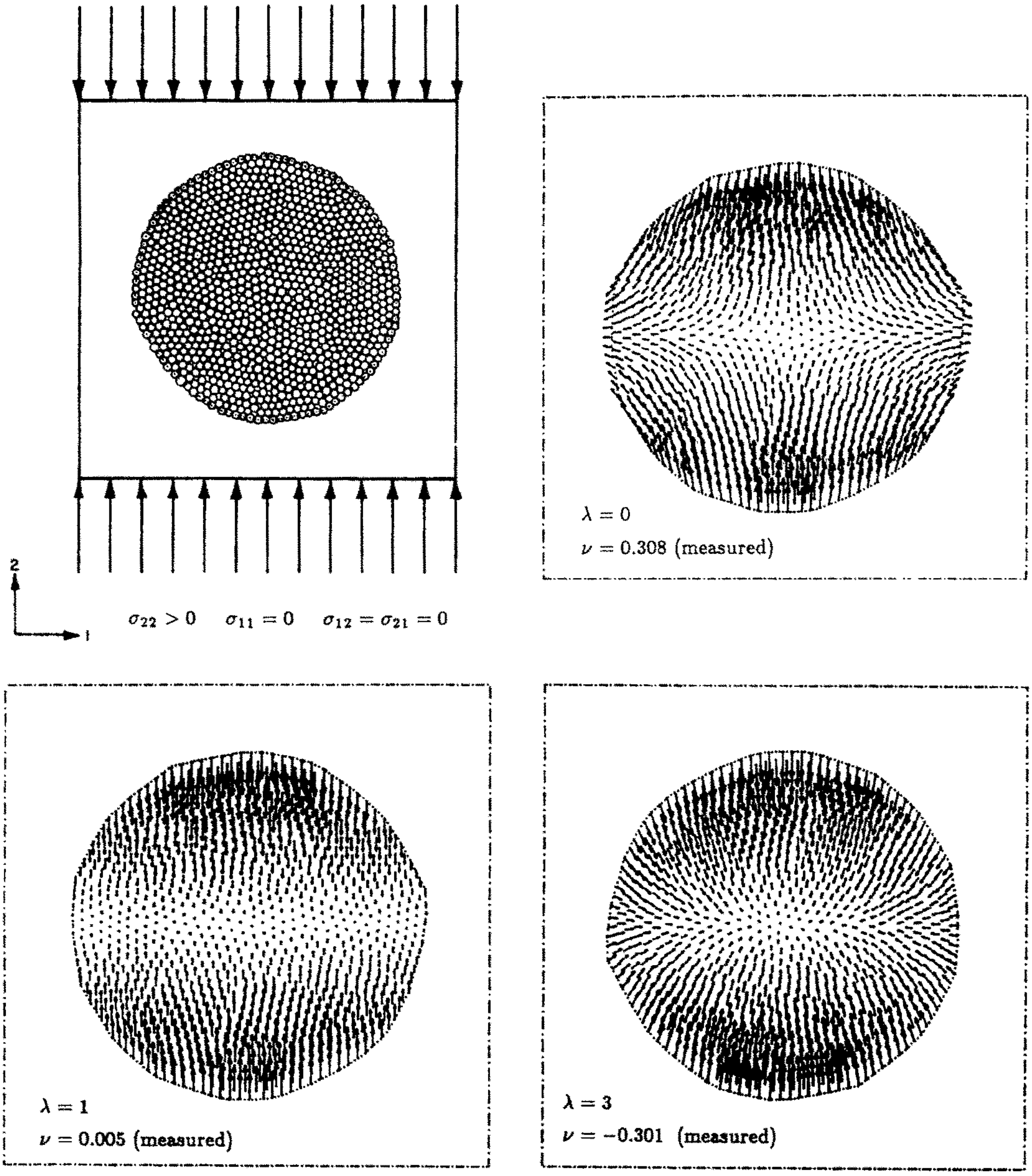

Fig. 7. Displacement fields for disc centres from biaxial compression tests with variable stiffness ratio λ.

Figure 7 shows particle centre displacement fields plotted from Poisson's ratio tests for assemblies with $\lambda = 0$, 1 and 3. The actual particle displacements have been greatly exaggerated since measured strains were typically $|\epsilon_{kk}| \leq 0.005$ in these tests. The plots show that the trends predicted by Poisson's ratio expression (14) are visually apparent in measured displacement fields. Of particular interest is the test with $\lambda = 3$ which shows that the predicted *negative* Poisson's ratio effect is indeed observed. The Poisson's ratio equation has been tested over the range $0 \leq \lambda \leq 3$ and the results presented on Fig. 8. The figure shows that there is good agreement between predicted and directly measured values of ν.

IMPLICATIONS TO THREE-DIMENSIONAL SYSTEMS

Numerical simulation of two-dimensional assemblies of discs can be thought of as an analogue to idealized assemblies of *bonded spheres* having variable radius and interacting through linear compliant contacts. Unfortunately, numerical simulation of these systems is

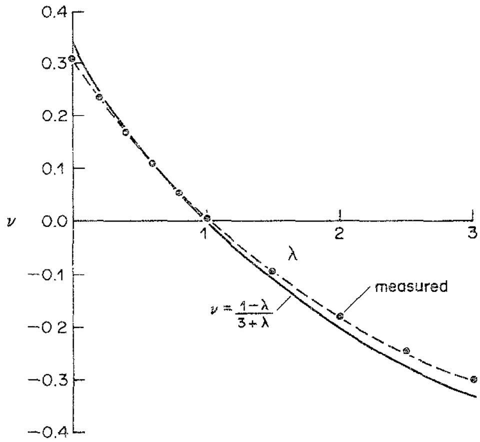

Fig. 8. Comparison of measured and theoretical values of Poisson's ratio ν against stiffness ratio λ.

prohibitively expensive for assemblies having a statistically meaningful number of particles. Nevertheless, the theoretical developments leading to the Poisson's ratio expression for two-dimensional systems are analogous to the approach which can be adopted to arrive at a similar expression for dense three-dimensional assemblies. For example, if three-dimensional assemblies of spheres with *isotropic* fabric are considered then $E(\Omega) = 1/4\pi$, $\bar{l}^c(\Omega) = \bar{l}_o$ and the stress tensor expression becomes:

$$\sigma_{ij} = \frac{m_v \bar{l}_o}{4\pi} \int_{\Omega} \bar{f}_i^c(\Omega) n_j(\Omega)\, \mathrm{d}\Omega \quad i, j = 1, 2, 3 \tag{16}$$

Here $\mathrm{d}\Omega = \sin\beta\, \mathrm{d}\beta\, \mathrm{d}\theta$ corresponding to the unit spherical coordinate system with $0 \le \beta \le \pi$ and $0 \le \theta \le 2\pi$. Components of the normal vector **n** are related to the unit spherical coordinate systems according to:

$$\begin{aligned} n_1 &= \sin\beta \sin\theta \\ n_2 &= \cos\beta \\ n_3 &= \sin\beta \cos\theta \end{aligned} \tag{17}$$

Rothenburg and Selvadurai [12] have shown that if particle rotations are negligible, average contact force components in relation (16) can be equated to the strain tensor according to:

$$\bar{f}_i^c(\Omega) = \zeta k_n \{\lambda \epsilon_{il} n_l + (1-\lambda)(\epsilon_{kl} n_l n_k) n_i\} \qquad i, k, l = 1, 2, 3 \tag{18}$$

Substitution of (17) and (18) into (16) leads to an expression for Poisson's ratio as follows:

$$\nu = \frac{1-\lambda}{4+\lambda} \tag{19}$$

Based on experience from numerical simulation of two-dimensional systems, it is reasonable to expect that this relationship is valid for dense assemblies of bonded spheres with linear contact interactions. If these assemblies are restricted to a narrow range of particle sizes then, dense assemblies would correspond to systems with $\gamma \to 12$.

According to Mindlin [15], the initial ratio of tangential to normal stiffness for isotropic elastic spheres in contact without slip varies over the range $0.5 \le \lambda \le 1$. This range corresponds to a macroscopic Poisson's ratio from eqn (19) of $0 \le \nu \le 0.11$. For perfect slip between spheres in contact, $\lambda = 0$ and eqn (19) gives $\nu = 1/4$ which is in agreement with Poisson for random assemblies of spheres with central interactions.

Equation (19) offers a micromechanical explanation of why granular assemblies with negative Poisson's ratio are not observed in nature. The requirement that particle contacts be stiffer in the tangential direction as compared to the normal direction is physically unlikely for particles comprised of natural materials.

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NOMENCLATURE

a, b	second and fourth-order coefficients of contact normal anisotropy
$E(\theta), E(\Omega)$	contact normal distribution function
$\mathbf{f}^c$	contact force vector
f_n^c, f_s^c	normal and tangential (shear) contact force components
$\bar{f}_n^c(\theta), \bar{f}_s^c(\theta)$	distribution of *average* normal and tangential (shear) contact forces
k_n, k_s	linear normal and tangential (shear) contact stiffnesses
l	distance between centres of particles in mutual contact
$\mathbf{l}^c$	contact vector
$\bar{l}^c(\theta), \bar{l}^c(\Omega)$	distribution of average contact lengths
$\bar{l}_0$	assembly average contact vector length
m_v	contact density $(= M_V/V)$
$M_g(\theta_g)$	number of contacts falling within interval θ_g
M_V	total number of assembly contacts
$\mathbf{n}^c$	contact normal vector
$n_i(\theta), n_i(\Omega)$	normal scalar components of unit vector
N	total number of assembly particles
$\mathbf{t}^c$	contact tangent vector
$t_j(\theta)$	tangential scalar components of unit vector
V	assembly area (or volume)
$\delta_n^c, \delta_t^c, \delta_w^c$	relative normal, translational shear and rotational shear contact displacement components
$\bar{\delta}_n^c(\theta), \bar{\delta}_t^c(\theta)$	distribution of *average* relative normal and translational shear contact displacements
ϵ, ϵ_{ij}	strain tensor
γ	coordination number $(= M_V/N)$
λ	ratio of contact stiffnesses $(= k_s/k_n)$
θ_a, θ_b	second and fourth-order principal directions of contact anisotropy
ν	Poisson's ratio
σ, σ_{ij}	stress tensor
$\bar{\sigma}, \bar{\sigma}_{ij}$	average stress tensor
ζ	constant relating average relative normal and tangential displacements to strain tensor